Non-Dimensionalisation of Navier-Stokes Equation for Cylindrical Coordinates System and Derivation using First Principles

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1 Introduction

This abstract is for the derivation for the non-dimensionalised Navier Stokes equation governing the cylindrical flow field.

2 Annotations Used

Serial No.	Annotation	Description
1	x	Cartesian Horizontal Axis/Distance
2	y	Cartesian Vertical Axis/Distance
3	z	Axial Direction (Cartesian + Cylindrical)
4	r	Radial Direction
5	θ	Azimuthal Direction
6	δz	Discretized Axial Length
7	δr	Discretized Radial Length
8	$\delta\theta$	Discretized Angular Distance
9	$\hat{i}$	Unit vector along X axis
10	$\hat{j}$	Unit vector along Y axis
11	$\hat{k}$	Unit vector along Z axis in Cartesian Coordinate
12	$\hat{e_r}$	Unit Vector along Radial Direction from central axis
13	$\hat{e_\theta}$	Unit Vector along Azimuthal Direction from XY Plane
14	$\hat{e_z}$	Unit Vector along Z axis in Cylindrical Coordinate
15	$\vec{p}$	Position Vector of a point or element in both Co-ordinates systems
16	$\vec{V}$	Net Velocity of a point or element in both Co-ordinates systems
17	u_z	Axial Velocity Magnitude
18	u_r	Radial Velocity Magnitude
19	u_θ	Azimuth Velocity Magnitude
20	P	Pressure
21	Re	Reynolds Number
22	t	time
23	δt	time change
24	ν	Kinematic Viscosity
25	μ	Dynamic Viscosity
26	ρ	Density
27	τ_{ij}	Shear Stress on i^{th} face in j^{th} direction
28	z	Axial Direction
29	F	Force Acting
30	SA	Surface Area of integration
31	CV	Control Volume
32	KN	Knudsen Number

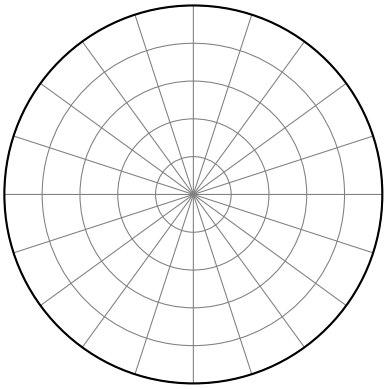
3 Derivation of the Governing Equation

In order to understand the physics behind the flow of the pipe problem we have to derive the mathematics of the governing equation which the fluid is obeying while flowing through the domain of interest.

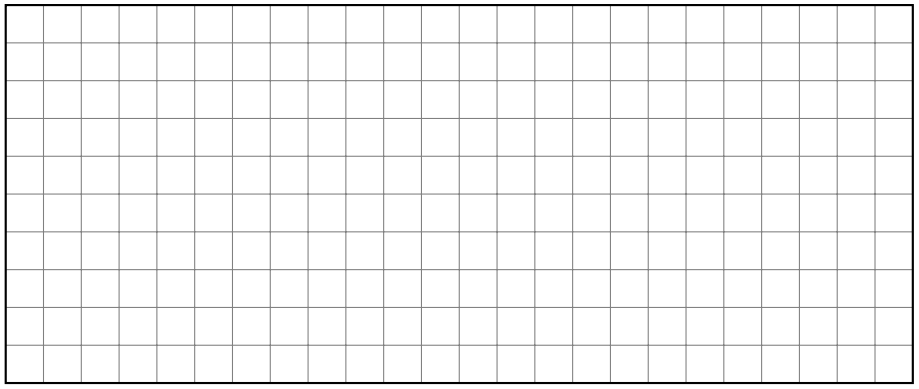
3.1 Assumptions for the derivation

1. The fluid is in-compressible and irrotational, i.e., ρ is constant with space and time throughout the flow field and the vorticity is 0, i.e. fluid particles within the control volume do not experience any rotational/turning effect.
2. The fluid behavior under the initial derivation is not specified to be of a particular type but towards the end of the derivation it will be chosen as Newtonian, that means it obeys the Newton's law of viscosity ,i.e., $\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$
3. The control volume is stationary and not accelerating, which also means that there is no gravity force acting which may influence the flow. The only external force acting is the pressure force and none other.
4. The flow is in continuum that means KN > 0.1.
5. Pressure acts only normal to a plane or surface compressible force and is the only external force acting.

3.2 Pictorial Representation of Flow Domain



(a) Front View of the Cylindrical Flow Domain



(b) Mid-Longitudinal Cross-Sectional View of the Cylindrical Flow Domain

Figure 1: Views for the discretized flow domain with various elements for the purpose of calculations for finite element modeling

Each cell in the domain represents a discretized element or volume as per say. In each volume the fluid particle is assumed to share some common properties like pressure and velocities along the three cylindrical coordinate vectors. Each of these cells are bounded by 6 surfaces namely top, bottom, left, right, front and back as per the viewers angle of view. As the pressure only acts normally while each face has 3 different components of the stresses acting along the 3 co-ordinate vectors namely :- $\hat{e_r}$, $\hat{e_\theta}$ & $\hat{e_z}$

3.3 Kinematic relation between Cartesian and Cylindrical co-ordinate systems

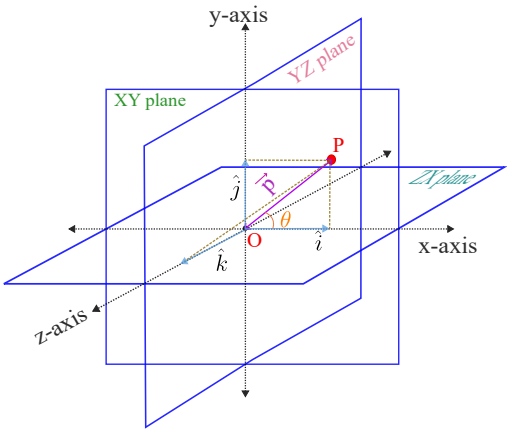
In this part we shall focus on the how the various relation between a Cartesian x,y,z system is helpful in deriving the polar r,θ, z system. The basic difference between a Cartesian and cylindrical/polar coordinate system happens to stem from the fact that the basis vectors have no variation in direction with time so they remain fixed throughout the calculation or formulation. Once they are defined with respect to the body they remain fixed with time and their orientation does not change but this does not happens with the polar coordinate as the two basis $\hat{e_r}$ & $\hat{e_\theta}$ rotate with time and keeps on changing. So for our starting we need a fixed absolute system which we can safely rely on the Cartesian for our initial derivation. So consider a **point P** located in space where a Cartesian system is well-defined that has its origin at **point O** and three perpendicular planes namely XY, YZ and ZX from whose mutual intersection brings about our three co-ordinate axis's namely x,y & z with respective unit vectors along these axis be $\hat{i}$, $\hat{j}$ & $\hat{k}$ respectively.So,

$$\vec{p} = x\hat{i} + y\hat{j} + z\hat{k}$$

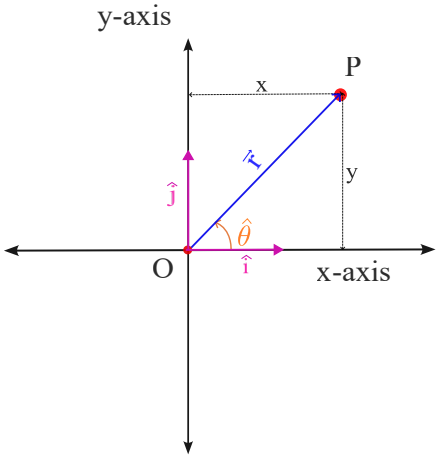
As we know that in Cartesian and polar co-ordinates the z axis remains the same just denoted by different basis as $\hat{k}$ in Cartesian and $\hat{e_z}$ in cylindrical. So we are more interested in the variation of XY and Rθ directions. So we can simply consider a parallel plane to XY which also contains point P and let $\vec{\mathbf{r}}$ be the vector pointing P originating from the z axis parallel to XY plane **r** be the magnitude of this vector. So,

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$=> \vec{r} = r\cos\theta\ \hat{i} + r\sin\theta\ \hat{j}$$



(a) 3D Cartesian Coordinate Geometry for Locating a Point and Initializing a Vector



(b) Relation Between Cartesian and Polar Coordinates

Figure 2: Comparison of Cartesian Coordinates and Polar Coordinates

$$=> \frac{\vec{r}}{r} = \cos\theta\ \hat{i} + \sin\theta\ \hat{j}$$

$$=> \hat{e_r} = \cos\theta\ \hat{i} + \sin\theta\ \hat{j}$$

This is the representation of radial unit vector and since axial unit vector is same as the z axis so it can be represented as $\hat{e_z} = \hat{k}$. Now in-order to find the azimuthal unit vector we will use the properties of the basis vectors. So, let us assume $\hat{e_\theta} = a\hat{i} + b\hat{j}$ First property is that in a system any two basis vectors must be orthonormal to each other which implies their dot product is zero:-

$$\hat{e_\theta}.\hat{e_r} = 0$$

$$=> (a\hat{i} + b\hat{j}).(\cos\theta\ \hat{i} + \sin\theta\ \hat{j}) = 0$$

$$=> a\cos\theta + b\sin\theta = 0$$

$$=> a = \frac{-b\sin\theta}{\cos\theta} \tag{1}$$

Second property is that the magnitude of a unit vector is always 1 that is :-

$$|a\hat{i} + b\hat{j}| = 1$$

$$a^2 + b^2 = 1$$

From equation 1 substitute a to get:

$$b^2\left(\frac{\sin^2\theta}{\cos^2\theta} + 1\right) = 1$$

$$=> b^2 = \cos^2\theta$$

$$=> b = \pm\cos\theta$$

Similarly,

$$=> a = \mp\sin\theta$$

So its is but our choice to choose $\pm$ or $\mp$ but choosing the first makes the angular basis anticlockwise as positive which follows the Right hand thumb rule for deciding directions for angular coordinates and is the General Convention. So we get our three basis vectors as follows :-

$$\hat{e_r} = \cos\theta\ \hat{i} + \sin\theta\ \hat{j} \quad , \quad \hat{e_\theta} = -\sin\theta\ \hat{i} + \cos\theta\ \hat{j} \quad , \quad e_z = \hat{k}$$

So,

$$\vec{p} = x\hat{i} + y\hat{j} + z\hat{k} = r(\cos\theta\ \hat{i} + \sin\theta\ \hat{j}) + z\hat{k} = r\hat{e_r} + z\hat{e_z}$$

Differentiating above equation with respect ti time we get :-

$$\vec{V_{net}} = \frac{d\vec{p}}{dt} = \frac{dr}{dt}\hat{e_r} + r\frac{d\hat{e_r}}{dt} + \frac{dz}{dt}\hat{e_z} + z\frac{d\hat{e_z}}{dt}$$

$$=> \vec{V_{net}} = u_r\hat{e_r} + r\frac{d\left(\cos\theta\ \hat{i} + \sin\theta\ \hat{j}\right)}{dt} + u_z\hat{e_z} + z\frac{d\hat{e_z}}{dt}$$

$$=> \vec{V_{net}} = u_r\hat{e_r} + r\left(-\sin\theta\frac{d\theta}{dt}\hat{i} + \cos\theta\frac{d\hat{i}}{dt} + \cos\theta\frac{d\theta}{dt}\hat{j} + \sin\theta\frac{d\hat{j}}{dt}\right) + u_z\hat{e_z} + z\frac{d\hat{e_z}}{dt}$$

As we know that $\hat{i}, \hat{j} \& \hat{k} = \hat{e_z}$ do not change with time so they are constants so $\frac{d\hat{i}}{dt} = \frac{d\hat{j}}{dt} = \frac{d\hat{k}}{dt} = \frac{d\hat{e_z}}{dt} = 0$

$$=> \vec{V_{net}} = u_r\hat{e_r} + r\frac{d\theta}{dt}\left(-\sin\theta\ \hat{i} + \cos\theta\ \hat{j}\right) + u_z\hat{e_z}$$

$$=> \vec{V_{net}} = u_r\hat{e_r} + u_\theta\hat{e_\theta} + u_z\hat{e_z} \tag{2}$$

There are some important conclusive equations from the above derivation :-

$$\frac{d\hat{e_r}}{dt} = \frac{d\left(\cos\theta\ \hat{i} + \sin\theta\ \hat{j}\right)}{dt} = \frac{d\theta}{dt}\left(-\sin\theta\ \hat{i} + \cos\theta\ \hat{j}\right) = \omega\hat{e_\theta} = \frac{u_\theta}{r}\hat{e_\theta}$$

And,

$$\frac{d\hat{e_\theta}}{dt} = \frac{d\left(-\sin\theta\ \hat{i} + \cos\theta\ \hat{j}\right)}{dt} = -\frac{d\theta}{dt}\left(\cos\theta\ \hat{i} + \sin\theta\ \hat{j}\right) = -\omega\hat{e_r} = -\frac{u_\theta}{r}\hat{e_r}$$

Now differentiating equation 2 with respect to time :-

$$\vec{acc_{net}} = \frac{d\vec{V_{net}}}{dt} = \frac{d\left(u_r\hat{e_r} + u_\theta\hat{e_\theta} + u_z\hat{e_z}\right)}{dt}$$

$$=> \vec{acc_{net}} = \frac{du_r}{dt}\hat{e_r} + u_r\frac{d\hat{e_r}}{dt} + \frac{du_\theta}{dt}\hat{e_\theta} + u_\theta\frac{d\hat{e_\theta}}{dt} + \frac{du_z}{dt}\hat{e_z} + u_z\frac{d\hat{e_z}}{dt}$$

$$=> \vec{acc_{net}} = \frac{du_r}{dt}\hat{e_r} + \frac{u_ru_\theta}{r}\hat{e_\theta} + \frac{du_\theta}{dt}\hat{e_\theta} - \frac{u_\theta^2}{r}\hat{e_r} + \frac{du_z}{dt}\hat{e_z}$$

$$=> \frac{d\vec{V_{net}}}{dt} = \vec{acc_{net}} = \left(\frac{du_r}{dt} - \frac{u_\theta^2}{r}\right)\hat{e_r} + \left(\frac{u_ru_\theta}{r} + \frac{du_\theta}{dt}\right)\hat{e_\theta} + \frac{du_z}{dt}\hat{e_z} \tag{3}$$

This is the relation of acceleration in a cylindrical co-ordinate system and will help us in our force analysis to get the dimensional cylindrical Navier Stokes equation.

4 Dynamic analysis of cylindrical flow field

Now we will subdivide the system into discrete elements or volumes for our calculations and analyses just one of these. An illustrated figure is shown depicting the various faces and a table shown to highlight the areas , flow velocity, pressure and stresses acting on various faces. Now we will consider the basic conservation laws of mass and momentum to derive our governing equations.

Face	**Area**	**Velocity**	**Pressure**	**Axial Stress**	**Radial Stress**	**Azimuthal Stress**
Back	$r\delta r\delta\theta$	u_z	P	τ_{zz}	τ_{zr}	$\tau_{z\theta}$
Front	$r\delta r\delta\theta$	$u_z + \frac{\partial u_z}{\partial z}\delta z$	$P + \frac{\partial P}{\partial z}\delta z$	$\tau_{zz} + \frac{\partial \tau_{zz}}{\partial z}\delta z$	$\tau_{zr} + \frac{\partial \tau_{zr}}{\partial z}\delta z$	$\tau_{z\theta} + \frac{\partial \tau_{z\theta}}{\partial z}\delta z$
Top	$(r+\delta r)\,\delta\theta\delta z$	$u_r + \frac{\partial u_r}{\partial r}\delta r$	$P + \frac{\partial P}{\partial r}\delta r$	$\tau_{rz} + \frac{\partial \tau_{rz}}{\partial r}\delta r$	$\tau_{rr} + \frac{\partial \tau_{rr}}{\partial r}\delta r$	$\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial r}\delta r$
Bottom	$r\delta\theta\delta z$	u_r	P	τ_{rz}	τ_{rr}	$\tau_{r\theta}$
Right	$\delta z\delta r$	u_θ	P	$\tau_{\theta z}$	$\tau_{\theta r}$	$\tau_{\theta\theta}$
Left	$\delta z\delta r$	$u_\theta + \frac{\partial u_\theta}{\partial \theta}\delta\theta$	$P + \frac{\partial P}{\partial \theta}\delta\theta$	$\tau_{\theta z} + \frac{\partial \tau_{\theta z}}{\partial \theta}\delta\theta$	$\tau_{\theta r} + \frac{\partial \tau_{\theta r}}{\partial \theta}\delta\theta$	$\tau_{\theta\theta} + \frac{\partial \tau_{\theta\theta}}{\partial \theta}\delta\theta$

Table 1: Flow parameters and stresses across different faces of the pipe.

4.1 Continuity Equation

In our assumption we state that the flow is in continuum and it allows us to use differential equations to help us guide to the actual governing equation. Also it is worth noting that the mass conservation law which states that **Mass can never be created nor be destroyed in classical perspective or in continuum hypothesis**. So, we deduce that:

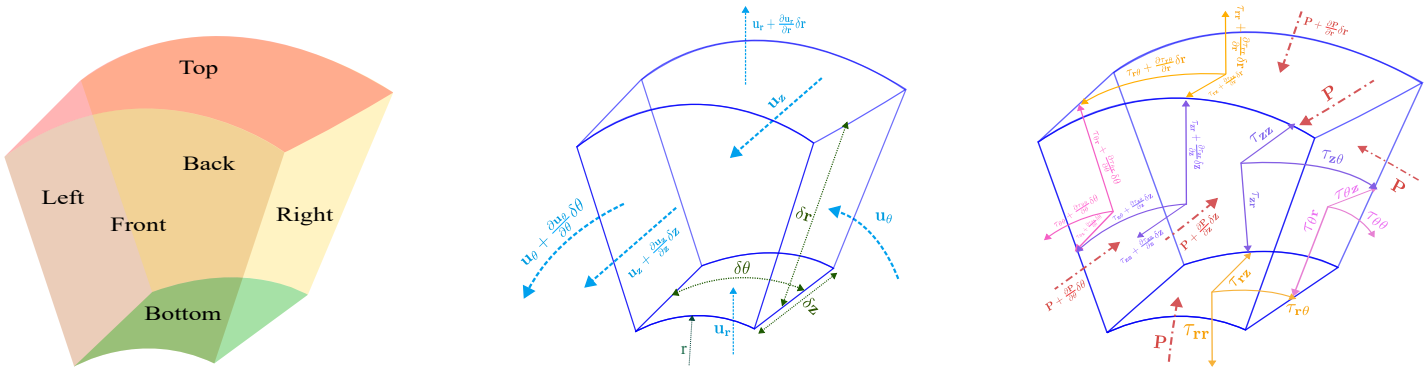
Mass entering the flow field = Mass leaving the flow field + Mass accumulated in the flow field

$$=> M_{inlet} = M_{outlet} + M_{stored}$$

Differentiating with respect to time the above equation we get:-

$$=> \frac{dM_{inlet}}{dt} = \frac{dM_{outlet}}{dt} + \frac{dM_{stored}}{dt}$$

$$=> \frac{d\left(\rho V_{inlet}\right)}{dt} = \frac{d\left(\rho V_{outlet}\right)}{dt} + \frac{d\left(\rho V_{stored}\right)}{dt}$$



(a) Polar element with all 6 faces (b) Velocities on various faces (c) Forces acting on all its faces

Figure 3: Different Aspects of a cylindrical element with mass flow and force analysis

Since the flow is in-compressible so canceling ρ from both the sides as well as since there is no change in control volume with time so , $\frac{dV_{stored}}{dt} = 0$, So.

$$=> \frac{dV_{inlet}}{dt} = \frac{dV_{outlet}}{dt}$$

As we can see from figure 3a and 3b that mass is entering from bottom right and front surface, while exiting from top, left and back surfaces; so.

$$\frac{dV_{front}}{dt} + \frac{dV_{bottom}}{dt} + \frac{dV_{right}}{dt} = \frac{dV_{back}}{dt} + \frac{dV_{top}}{dt} + \frac{dV_{left}}{dt}$$

As the cross-sectional area of surfaces is independent of time, so only the flow length is affected which converts to velocity on differentiating:-

$$u_{front}A_{front} + u_{bottom}A_{bottom} + u_{right}A_{right} = u_{back}A_{back} + u_{top}A_{top} + u_{left}A_{left}$$

$$=> u_z.r\delta r\delta\theta + u_r.r\delta\theta\delta z + u_\theta.\delta r\delta z = \left(u_z + \frac{\partial u_z}{\partial z}\delta z\right).r\delta r\delta\theta + \left(u_r + \frac{\partial u_r}{\partial r}\delta r\right).(r+\delta r)\,\delta\theta\delta z + \left(u_\theta + \frac{\partial u_\theta}{\partial\theta}\delta\theta\right).\delta r\delta z$$

$$=>u_z r\delta r\delta\theta + u_r.r\delta\theta\delta z + u_\theta.\delta r\delta z = u_z r\delta r\delta\theta + \frac{\partial u_z}{\partial z}\delta z r\delta r\delta\theta + u_r r\delta\theta\delta z + u_r\delta r\delta\theta\delta z + \frac{\partial u_r}{\partial r}\delta r r\delta\theta\delta z + \frac{\partial u_r}{\partial r}\delta r\delta r\delta\theta\delta z +$$
$$u_\theta\delta r\delta z + \frac{\partial u_\theta}{\partial\theta}\delta\theta\delta r\delta z$$

Canceling the common terms we get :-

$$0 = \frac{\partial u_z}{\partial z}r\delta z\delta r\delta\theta + u_r\delta r\delta\theta\delta z + \frac{\partial u_r}{\partial r}r\delta r\delta\theta\delta z + \frac{\partial u_r}{\partial r}\delta r\delta r\delta\theta\delta z + \frac{\partial u_\theta}{\partial\theta}\delta\theta\delta r\delta z$$

Dividing the equation by $\delta z\delta r\delta\theta$ we get:-

$$r\frac{\partial u_z}{\partial z} + u_r + r\frac{\partial u_r}{\partial r} + \partial r\frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial\theta} = 0$$

Important to note :-

$$\frac{\partial(ru_r)}{\partial r} = u_r + \frac{\partial u_r}{\partial r} \qquad \& \qquad \partial r \approx 0$$

So,

$$\implies \frac{\partial(ru_r)}{\partial r} + \frac{\partial u_\theta}{\partial\theta} + r\frac{\partial u_z}{\partial z} = 0$$

Dividing the above equation by r we get :-

$$\boxed{\implies \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z} = 0} \tag{4}$$

The above equation is known as the **Continuity Equation for In-compressible Flows for Cylindrical Co-ordinates**

4.2 The Three Momentum Equations

Before diving into the momentum equation we must first revise the three **NEWTONIAN LAWS** for classical mechanics :-

1. Law 1 - An object at rest will remain at rest, and an object in motion will continue in a straight line with constant speed, unless acted upon by a net external force.

2. Law 2 - The acceleration of an object is directly proportional to the net force acting on the object and inversely proportional to its mass.

3. Law 3 - For every action, there is an equal and opposite reaction.

These laws form the very base of the governing equation for the basic fluid flow problems. By understanding the first law as the fluid is at equilibrium - stationary of moving with a constant Cartesian velocity it keeps on maintaining its original state when no external force is acting which gives us over **Initial Condition** for the flow field. Now just after a short duration of time an initial inlet pressure starts to act which forces or pushes the fluid forward and causes the change in velocity, causing changes in the fluid regime. Upon external forces acting on the fluid body it tries to withstand or resist this pressure force which develops some internal stresses denoted by $\tau_{\mathbf{ij}}$ on various surfaces. Notably the normal components of these force resist the pressure which is naturally a compressing force, so these internal resistances are tensile in nature and act outward from the body of the fluid. Meanwhile the tangential or shear component of the force resist surface deformation as well as the angular changes in velocity of the fluid. Collectively it is this communication between internal and external forces which causes change in motion or velocities of the fluid particle and gets the engineers involved in their study.
So, our main points from the discussion were:-

- **PRESSURE** is the external hydro-static force or main cause of change in motion or inertia of the fluid. It is compressive on all faces it is acting.
- Stresses are internal resistive forces called as **VISCOUS FORCE** that are reaction to the pressure forces and resist the motion of fluid. It is of two types - Normal and Shear. while only a single normal stress acts in outward direction the shear stresses always act along or tangential to the surface in pair to prevent the rotational effect of forces on the fluid element.
- Additionally some **CURVATURE** force will also be present due to the angular path not being straight and causes an additional hindrance to the inertia of the fluid particle trying to move along a straight line. Interestingly these forces will only affect the radial and azimuthal direction and not the axial direction.
- **It is but most important to note that while finding out the pressure and viscous forces only the translation components,i.e., along radial and axial directions are considered and not azimuthal. Its is plainly because the azimuthal force analysis not only depends on the magnitude and direction of forces but also on the Distance of the force from the central axis of rotation which is called as MOMENT OF FORCE and it is completely covered in CURVATURE Force analysis.**
- The net affect of all forces causes the fluid element / body to develop some acceleration along flow field.

$$\vec{F_{net}} = \vec{F_{pressure}} + \vec{F_{viscous}} + \vec{F_{curvature}}$$

So, by using the Newton's second law of motion, we get :-

$$\vec{F_{net}} = \frac{d(m\vec{V_{net}})}{dt}$$

$$\implies \frac{d(m\vec{V_{net}})}{dt} = \vec{F_{pressure}} + \vec{F_{viscous}} + \vec{F_{curvature}} \tag{5}$$

From figure (3c) and table (1) we can deduce the pressure and viscous forces :-

$$\vec{F_{pressure}} = \left(P - \left(P + \frac{\partial P}{\partial z}\delta z\right)\right) r\delta r\delta\theta\hat{e_z} + \left(Pr\delta\theta\delta z - \left(P + \frac{\partial P}{\partial r}\delta r\right)(r+\delta r)\,\delta z\delta\theta\right)\hat{e_r}$$

$$\implies \vec{F_{pressure}} = -\frac{\partial P}{\partial z} r\delta r\delta\theta\delta z\hat{e_z} - \left(P\delta r\delta z\delta\theta + \frac{\partial P}{\partial r} r\delta r\delta z\delta\theta + \frac{\partial P}{\partial r}\delta r^2\delta z\delta\theta\right)\hat{e_r}$$

$$\implies \vec{F_{pressure}} = -\frac{\partial P}{\partial z} r\delta r\delta\theta\delta z\hat{e_z} - \left(\frac{P}{r} + \frac{\partial P}{\partial r} + \frac{\partial P}{\partial r}\frac{\delta r}{r}\right) r\delta r\delta z\delta\theta\hat{e_r}$$

As we can see along the radial basis : as $\delta r \approx 0$ so effect of $\frac{\mathbf{P}}{\mathbf{r}} + \frac{\partial \mathbf{P}}{\partial \mathbf{r}} >> \frac{\partial \mathbf{P}}{\partial \mathbf{r}}\frac{\delta \mathbf{r}}{\mathbf{r}}$. So,

$$\implies \vec{F_{pressure}} = -\left(\frac{P}{r} + \frac{\partial P}{\partial r}\right) r\delta r\delta z\delta\theta\hat{e_r} - \frac{\partial P}{\partial z} r\delta r\delta\theta\delta z\hat{e_z} \tag{6}$$

$$\implies \vec{F_{viscous}} = \Bigg(\left(\tau_{rr} + \frac{\partial \tau_{rr}}{\partial r}\delta r\right)(r+\delta r)\,\delta\theta\delta z + \left(\tau_{\theta r} + \frac{\partial \tau_{\theta r}}{\partial \theta}\delta\theta\right)\delta z\delta r + \left(\tau_{zr} + \frac{\partial \tau_{zr}}{\partial r}\delta z\right) r\delta\theta\delta r - \tau_{rr} r\delta\theta\delta z - \tau_{zr} r\delta\theta\delta r$$
$$- \tau_{\theta r}\delta r\delta z\Bigg)\hat{e_r} + \Bigg(\left(\tau_{zz} + \frac{\partial \tau_{zz}}{\partial z}\delta z\right) r\delta r\delta\theta + \left(\tau_{rz} + \frac{\partial \tau_{rz}}{\partial r}\delta r\right)(r+\delta r)\,\delta\theta\delta z + \left(\tau_{\theta z} + \frac{\partial \tau_{\theta z}}{\partial \theta}\delta\theta\right)\delta z\delta r - \tau_{zz} r\delta\theta\delta r - \tau_{\theta z}\delta z\delta r\Bigg)\hat{e_z}$$

$$\implies \vec{F_{viscous}} = \left(\tau_{rr}\delta r\delta\theta\delta z + r\frac{\partial \tau_{rr}}{\partial r}\delta r\delta\theta\delta z + \frac{\partial \tau_{rr}}{\partial r}\delta r^2\delta\theta\delta z + \frac{\partial \tau_{\theta r}}{\partial \theta}\delta\theta\delta z\delta r + \frac{\partial \tau_{zr}}{\partial r} r\delta\theta\delta r\delta z\right)\hat{e_r} +$$
$$\left(r\frac{\partial \tau_{zz}}{\partial z}\delta z\delta r\delta\theta + \tau_{rz}\delta r\delta\theta\delta z + r\frac{\partial \tau_{rz}}{\partial r}\delta r\delta\theta\delta z + \frac{\partial \tau_{rz}}{\partial r}\delta r^2\delta\theta\delta z + \frac{\partial \tau_{\theta z}}{\partial \theta}\delta\theta\delta z\delta r\right)\hat{e_z}$$

Since we know that $\delta r \to 0$ so $\delta r^2 \approx 0$ as well as an extra derivative will not even get canceled making the derivative even smaller, and also tking the common terms outside brackets we get :-

$$\implies \vec{F_{viscous}} = \left(\left(\tau_{rr} + r\frac{\partial \tau_{rr}}{\partial r}\right) + \frac{\partial \tau_{\theta r}}{\partial \theta} + r\frac{\partial \tau_{zr}}{\partial r}\right)\delta r\delta\theta\delta z\hat{e_r} + \left(\left(\tau_{rz} + r\frac{\partial \tau_{rz}}{\partial r}\right) + \frac{\partial \tau_{\theta z}}{\partial \theta} + r\frac{\partial \tau_{zz}}{\partial z}\right)\delta r\delta\theta\delta z\hat{e_z}$$

$$\implies \vec{F_{viscous}} = \left(\frac{\partial (r\tau_{rr})}{\partial r} + \frac{\partial \tau_{\theta r}}{\partial \theta} + r\frac{\partial \tau_{zr}}{\partial r}\right)\delta r\delta\theta\delta z\hat{e_r} + \left(\frac{\partial (r\tau_{rz})}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial \theta} + r\frac{\partial \tau_{zz}}{\partial z}\right)\delta r\delta\theta\delta z\hat{e_z} \tag{7}$$

In order to determine the curvature effect we will look at radial and azimuthal stress analysis and find out the additional forces. In radial we can easily make out that due to the turning effect, the normal azimuthal force is having a radial component also which is acting like a centripetal force (center driving force causing rotary motion) so it will be added. Further, in the azimuthal direction, in addition to the forces there is also the moment of forces which needs to be balanced as well and will be converted to azimuthal force alongside it.
In order to determine the curvature effect we will look at radial and azimuthal stress analysis and find out the additional forces. In radial we can easily make out that due to the turning effect, the normal azimuthal force is having a radial component also which is acting like a centrifugal force so it will be added. Further, in the azimuthal direction, in addition to the forces there is also the moment of forces which needs to be balanced as well and will be converted to azimuthal force alongside it.

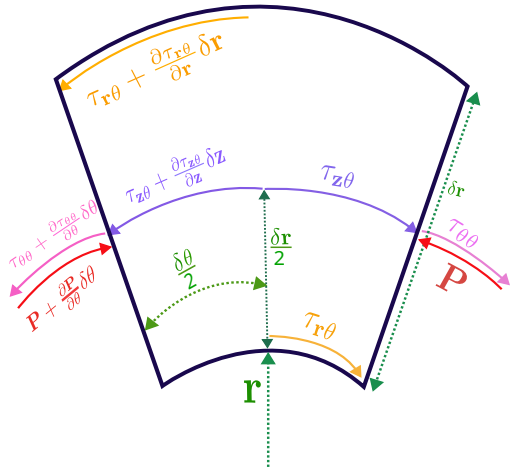


Figure 4: Illustration of the Curvilinear Forces on the element

$$\vec{F_{curvature}} = \left(-\tau_{\theta\theta}\sin\left(\frac{\partial\theta}{2}\right)\partial r\partial z - \left(\tau_{\theta\theta} + \frac{\partial \tau_{\theta\theta}}{\partial\theta}\delta\theta\right)\sin\left(\frac{\partial\theta}{2}\right)\partial r\partial z + P\sin\left(\frac{\delta\theta}{2}\right)\delta r\delta z + \left(P + \frac{\partial P}{\partial\theta}\delta\theta\right)\sin\left(\frac{\delta\theta}{2}\right)\delta r\delta z\right)\hat{e_r}$$
$$+ \left(\frac{\begin{array}{c}\left(\left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial r}\delta r\right)(r+\delta r)^2\delta\theta\delta z - \tau_{r\theta}r^2\delta\theta\delta z + \left(\tau_{z\theta} + \frac{\partial \tau_{z\theta}}{\partial z}\delta z\right)r\delta\theta\delta r\left(r + \frac{\delta r}{2}\right) - \tau_{z\theta}r\delta\theta\delta r\left(r+\frac{\delta r}{2}\right) + \right.\\ \left.\left(\tau_{\theta\theta} + \frac{\partial \tau_{\theta\theta}}{\partial\theta}\delta\theta\right)\delta z\delta r\left(r+\frac{\delta r}{2}\right) - \tau_{\theta\theta}\delta z\delta r\left(r+\frac{\delta r}{2}\right) - \left(P + \frac{\partial P}{\partial\theta}\delta\theta\right)\delta z\delta r\left(r+\frac{\delta r}{2}\right) + P\delta z\delta r\left(r+\frac{\delta r}{2}\right)\right)\end{array}}{r + \frac{\delta r}{2}}\right)\hat{e_\theta}$$

$$\because \delta\theta \to 0 \implies \sin\left(\frac{\delta\theta}{2}\right) = \frac{\delta\theta}{2} \qquad \& \qquad \frac{\delta r}{2} << r \text{ So we get that;}$$

$$\implies \vec{F_{curvature}} = \left(-\tau_{\theta\theta}\frac{\partial\theta}{2} - \tau_{\theta\theta}\frac{\partial\theta}{2} - \frac{\partial \tau_{\theta\theta}}{\partial\theta}\delta\theta\frac{\partial\theta}{2} + P\frac{\delta\theta}{2} + P\frac{\delta\theta}{2} + \frac{\partial P}{\partial\theta}\delta\theta\frac{\delta\theta}{2}\right)\partial r\partial z\hat{e_r} +$$
$$\left(\frac{\begin{array}{c}\left(\left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial r}\delta r\right)\left(r^2 + 2r\delta r + \delta r^2\right)\delta\theta\delta z - \tau_{r\theta}r^2\delta\theta\delta z + \frac{\partial \tau_{z\theta}}{\partial z}\delta z r\delta\theta\delta r\left(r+\frac{\delta r}{2}\right) + \right.\\ \left.\frac{\partial \tau_{\theta\theta}}{\partial\theta}\delta\theta\delta z\delta r\left(r+\frac{\delta r}{2}\right) - \frac{\partial P}{\partial\theta}\delta\theta\delta z\delta r\left(r+\frac{\delta r}{2}\right)\right)\end{array}}{r + \frac{\delta r}{2}}\right)\hat{e_\theta}$$

$$\implies \vec{F_{curvature}} = \left(-\tau_{\theta\theta}\partial\theta + P\delta\theta - \frac{\partial \tau_{\theta\theta}}{\partial\theta}\frac{\partial\theta^2}{2} + \frac{\partial P}{\partial\theta}\frac{\partial\theta^2}{2}\right)\partial r\partial z\hat{e_r} +$$
$$\left(\frac{\left(\tau_{r\theta}\left(2r+\delta r\right)\delta r\delta\theta\delta z + r^2\frac{\partial \tau_{r\theta}}{\partial r}\delta r\delta\theta\delta z + 2r\frac{\partial \tau_{r\theta}}{\partial r}\delta r^2\delta\theta\delta z + \frac{\partial \tau_{r\theta}}{\partial r}\delta r^3\delta\theta\delta z\right)}{r + \frac{\delta r}{2}} + \frac{\partial \tau_{z\theta}}{\partial z}\delta z r\delta\theta\delta r + \frac{\partial \tau_{\theta\theta}}{\partial\theta}\delta\theta\delta z\delta r - \frac{\partial P}{\partial\theta}\delta\theta\delta z\delta r\right)$$

Neglecting higher powers of $\delta\theta, \delta r \& \delta z$ and taking all the common terms outside brackets we get :-

$$\implies \vec{F_{curvature}} = \left(-\tau_{\theta\theta} + P\right)\partial\theta\partial r\partial z\hat{e_r} + \left(2\tau_{r\theta} + \frac{\left(r+\delta r\right)^2\frac{\partial \tau_{r\theta}}{\partial r}}{r + \frac{\delta r}{2}} + r\frac{\partial \tau_{z\theta}}{\partial z} + \frac{\partial \tau_{\theta\theta}}{\partial\theta} - \frac{\partial P}{\partial\theta}\right)\partial\theta\partial r\partial z\hat{e_\theta}$$

As we know that $\delta r << r$, So we can neglect δr in addition with r and so by making some diligent multiplications we get :-

$$\implies \vec{F_{curvature}} = (-\tau_{\theta\theta} + P)\,\partial\theta\partial r\partial z\hat{e_r} + \left(\frac{1}{r}\left(2r\tau_{r\theta} + r^2\frac{\partial\tau_{r\theta}}{\partial r}\right) + r\frac{\partial\tau_{z\theta}}{\partial z} + \frac{\partial\tau_{\theta\theta}}{\partial\theta} - \frac{\partial P}{\partial\theta}\right)\partial\theta\partial r\partial z\hat{e_\theta}$$

Combibing te dreivative terms where possible and also taking r also common from the inner terms to outside brackets as done for other forces we get:-

$$\implies \vec{F_{curvature}} = \left(-\frac{\tau_{\theta\theta}}{r} + \frac{P}{r}\right)r\partial\theta\partial r\partial z\hat{e_r} + \left(\frac{1}{r^2}\frac{\partial\left(r^2\tau_{r\theta}\right)}{\partial r} + \frac{\partial\tau_{z\theta}}{\partial z} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} - \frac{1}{r}\frac{\partial P}{\partial\theta}\right)r\partial\theta\partial r\partial z\hat{e_\theta} \tag{8}$$

So, putting equations (6), (7) and (8) in (5), we get :-

$$\vec{F_{net}} = \vec{F_{pressure}} + \vec{F_{viscous}} + \vec{F_{curvature}}$$

$$\implies \frac{d\left(m\vec{V_{net}}\right)}{dt} = -\left(\frac{P}{r} + \frac{\partial P}{\partial r}\right)r\delta r\delta z\delta\theta\hat{e_r} - \frac{\partial P}{\partial z}r\delta r\delta\theta\delta z\hat{e_z} + \left(\frac{\partial(r\tau_{rr})}{\partial r} + \frac{\partial\tau_{\theta r}}{\partial\theta} + r\frac{\partial\tau_{zr}}{\partial r}\right)\delta r\delta\theta\delta z\hat{e_r} +$$
$$\left(\frac{\partial(r\tau_{rz})}{\partial r} + \frac{\partial\tau_{\theta z}}{\partial\theta} + r\frac{\partial\tau_{zz}}{\partial z}\right)\delta r\delta\theta\delta z\hat{e_z} + \left(-\frac{\tau_{\theta\theta}}{r} + \frac{P}{r}\right)r\partial\theta\partial r\partial z\hat{e_r} + \left(\frac{1}{r^2}\frac{\partial\left(r^2\tau_{r\theta}\right)}{\partial r} + \frac{\partial\tau_{z\theta}}{\partial z} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} - \frac{1}{r}\frac{\partial P}{\partial\theta}\right)r\partial\theta\partial r\partial z\hat{e_\theta}$$

Splitting the left side and arranging the common basis terms we get :-

$$\implies \vec{V_{net}}\frac{dm}{dt} + m\frac{d\vec{V_{net}}}{dt} = \left(-\frac{P}{r} - \frac{\partial P}{\partial r} + \frac{1}{r}\frac{\partial(r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta} + \frac{\partial\tau_{zr}}{\partial r} - \frac{\tau_{\theta\theta}}{r} + \frac{P}{r}\right)r\delta r\delta z\delta\theta\hat{e_r} +$$
$$\left(-\frac{\partial P}{\partial z} + \frac{1}{r}\frac{\partial(r\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{\partial\tau_{zz}}{\partial z}\right)r\delta r\delta\theta\delta z\hat{e_z} + \left(\frac{1}{r^2}\frac{\partial\left(r^2\tau_{r\theta}\right)}{\partial r} + \frac{\partial\tau_{z\theta}}{\partial z} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} - \frac{1}{r}\frac{\partial P}{\partial\theta}\right)r\partial\theta\partial r\partial z\hat{e_\theta}$$

As we know that, mass within the CV is constant so, $\frac{dm}{dt} = 0$ also that mass of fluid within the CV is given by the expression : $m = \rho r\delta\theta\delta r\delta z$. Hence using this expression for m as well as picking the expression for $\frac{\vec{V_{net}}}{dt}$ from equation (3), we get:-

$$\implies \rho r\delta\theta\delta r\delta z\left(\left(\frac{du_r}{dt} - \frac{u_\theta^2}{r}\right)\hat{e_r} + \left(\frac{u_r u_\theta}{r} + \frac{du_\theta}{dt}\right)\hat{e_\theta} + \frac{du_z}{dt}\hat{e_z}\right) = \left(-\frac{P}{r} - \frac{\partial P}{\partial r} + \frac{1}{r}\frac{\partial(r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta} + \frac{\partial\tau_{zr}}{\partial r} - \frac{\tau_{\theta\theta}}{r} + \frac{P}{r}\right)r\delta r\delta z\delta\theta\hat{e_r}$$
$$+\left(\frac{1}{r^2}\frac{\partial\left(r^2\tau_{r\theta}\right)}{\partial r} + \frac{\partial\tau_{z\theta}}{\partial z} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} - \frac{1}{r}\frac{\partial P}{\partial\theta}\right)r\partial\theta\partial r\partial z\hat{e_\theta} + \left(-\frac{\partial P}{\partial z} + \frac{1}{r}\frac{\partial(r\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{\partial\tau_{zz}}{\partial z}\right)r\delta r\delta\theta\delta z\hat{e_z}$$

Canceling $r\delta\theta\delta r\delta z$ from both sides, we get :-

$$\implies \rho\left(\left(\frac{du_r}{dt} - \frac{u_\theta^2}{r}\right)\hat{e_r} + \left(\frac{u_r u_\theta}{r} + \frac{du_\theta}{dt}\right)\hat{e_\theta} + \frac{du_z}{dt}\hat{e_z}\right) = \left(-\frac{\partial P}{\partial r} + \frac{1}{r}\frac{\partial(r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta} + \frac{\partial\tau_{zr}}{\partial r} - \frac{\tau_{\theta\theta}}{r}\right)\hat{e_r} \tag{9}$$
$$+\left(\frac{1}{r^2}\frac{\partial\left(r^2\tau_{r\theta}\right)}{\partial r} + \frac{\partial\tau_{z\theta}}{\partial z} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} - \frac{1}{r}\frac{\partial P}{\partial\theta}\right)\hat{e_\theta} + \left(-\frac{\partial P}{\partial z} + \frac{1}{r}\frac{\partial(r\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{\partial\tau_{zz}}{\partial z}\right)\hat{e_z}$$

This is the exact governing equation for an incompressible cylindrical pipe flow. Now we are going to focus on the total derivative of velocities with respect to time: Using the multi-variable chain rule for partial derivative dependencies :- Total Derivative of a function = Sum of partial derivatives of the function with respect to all dependent parameters.

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

Using the same principle on all the three velocity components in the cylindrical co-ordinate system we get :-

$$\frac{du_r}{dt} = \frac{\partial u_r}{\partial t} + \frac{\partial u_r}{\partial r}\frac{dr}{dt} + \frac{\partial u_r}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial u_r}{\partial z}\frac{dz}{dt}$$

$$\implies \frac{du_r}{dt} = \frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} + u_z\frac{\partial u_r}{\partial z} \tag{10}$$

Similarly, for other components as well :-

$$\implies \frac{du_\theta}{dt} = \frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta} + u_z\frac{\partial u_\theta}{\partial z} \tag{11}$$

$$\implies \frac{du_z}{dt} = \frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta} + u_z\frac{\partial u_z}{\partial z} \tag{12}$$

Now let us split the left and right sides of equation(9) in various independent basis vectors to get three momentum equations out of them as well as putting the relationships of equations - (10), (11) & (12) we get :-

1. **Radial Momentum Equation**

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\theta^2}{r}\right)=-\frac{\partial P}{\partial r}+\frac{1}{r}\frac{\partial(r\tau_{rr})}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta}+\frac{\partial \tau_{zr}}{\partial r}-\frac{\tau_{\theta\theta}}{r} \tag{13}$$

2. **Azimuthal Momentum Equation**

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+u_z\frac{\partial u_\theta}{\partial z}+\frac{u_r u_\theta}{r}\right)=-\frac{1}{r}\frac{\partial P}{\partial \theta}+\frac{1}{r^2}\frac{\partial\left(r^2\tau_{r\theta}\right)}{\partial r}+\frac{\partial \tau_{z\theta}}{\partial z}+\frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} \tag{14}$$

3. **Axial Momentum Equation**

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial P}{\partial z}+\frac{1}{r}\frac{\partial(r\tau_{rz})}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \tau_{zz}}{\partial z} \tag{15}$$

These are the governing equations for any in-compressible flow of fluids in an cylindrical geometry like pipe flows. These can be very likely verified from the book - **"Boundary-Layer Theory" by Hermann Schlichting** Chapter - 3.13 Page 79 [4]; in which although the the major derivation is not discussed but it highlights these as the main governing equations for the cylindrical co-ordinate systems. Now we will shift our focus to the Newtonian Fluid Flows. According to Newton's Law of Viscosity,
Shear Stress on a fluid $\propto$ Shear Strain rate of the surface on which the stress is acting, i.e. :-

$$\tau_{ij}=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$$

This equation is generally preferred for Cartesian coordinate as all basis vectors are independent of each other. But for a cylindrical system, it would be needed to be modified as explained using basic principles.

4.3 Newton's Law of Viscosity for Cylindrical Basis System

The figure (5) towards right illustrates the various velocity components acting on the differential fluid element whose dimensions are assumed to be ∂r, $r\partial\theta$, ∂z. The velocities are assumed to act about the central point of the element in their respective basis vector directions as $\hat{e_r}, \hat{e_\theta}\&\hat{e_z}$. Using the Taylor's law of series expansion for finding these velocities on all surfaces and neglecting the higher order terms we get :-

Element Face	Radial Velocity	Azimuthal Velocity	Axial Velocity
Top	$u_r+\frac{\partial u_r}{\partial r}\frac{\delta r}{2}$	$u_\theta+\frac{\partial u_\theta}{\partial r}\frac{\delta r}{2}$	$u_z+\frac{\partial u_z}{\partial r}\frac{\delta r}{2}$
Bottom	$u_r-\frac{\partial u_r}{\partial r}\frac{\delta r}{2}$	$u_\theta-\frac{\partial u_\theta}{\partial r}\frac{\delta r}{2}$	$u_z-\frac{\partial u_z}{\partial r}\frac{\delta r}{2}$
Left	$u_r+\frac{\partial u_r}{\partial \theta}\frac{\delta \theta}{2}$	$u_\theta+\frac{\partial u_\theta}{\partial \theta}\frac{\delta \theta}{2}$	$u_z+\frac{\partial u_z}{\partial \theta}\frac{\delta \theta}{2}$
Right	$u_r-\frac{\partial u_r}{\partial \theta}\frac{\delta \theta}{2}$	$u_\theta-\frac{\partial u_\theta}{\partial \theta}\frac{\delta \theta}{2}$	$u_z-\frac{\partial u_z}{\partial \theta}\frac{\delta \theta}{2}$
Front	$u_r+\frac{\partial u_r}{\partial z}\frac{\delta z}{2}$	$u_\theta+\frac{\partial u_\theta}{\partial z}\frac{\delta z}{2}$	$u_z+\frac{\partial u_z}{\partial z}\frac{\delta z}{2}$
Back	$u_r-\frac{\partial u_r}{\partial z}\frac{\delta z}{2}$	$u_\theta-\frac{\partial u_\theta}{\partial z}\frac{\delta z}{2}$	$u_z-\frac{\partial u_z}{\partial z}\frac{\delta z}{2}$

Table 2: Table specifying the velocity components on various faces

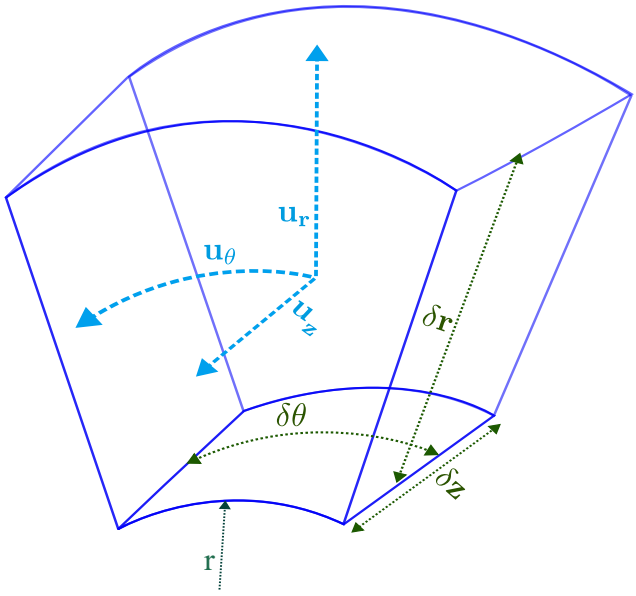


Figure 5: Elemental figure to support the derivation

Now with the help of this table we are going to find out relation for all stress components :-

$\tau_{rr}=2\mu\times$ Rate of shear strain of Mid-Radial length of the element along Radial Direction

$$\implies \tau_{rr}=2\mu\left(\frac{\left(u_r+\frac{\partial u_r}{\partial r}\frac{\delta r}{2}-\left(u_r-\frac{\partial u_r}{\partial r}\frac{\delta r}{2}\right)\right)}{\delta r}\right)$$

$$\implies \tau_{rr}=2\mu\frac{\partial u_r}{\partial r}$$

$\tau_{\theta\theta}=2\mu\times$ Rate of shear strain of Mid-Azimuthal length of the element along Azimuthal Direction

$$\implies \tau_{\theta\theta}=2\mu\left(\frac{\frac{\partial(r\delta\theta)}{\partial t}}{(r+\frac{\delta r}{2})\delta\theta}\right)=2\mu\left(\frac{r\frac{\partial(\delta\theta)}{\partial t}+\delta\theta\frac{\partial r}{\partial t}}{(r+\frac{\delta r}{2})\delta\theta}\right)$$

$$\implies \tau_{\theta\theta}=2\mu\left(\frac{\left(u_\theta+\frac{\partial u_\theta}{\partial \theta}\frac{\delta\theta}{2}-\left(u_\theta-\frac{\partial u_\theta}{\partial \theta}\frac{\delta\theta}{2}\right)\right)}{(r+\frac{\delta r}{2})\delta\theta}+\frac{u_r}{r+\frac{\delta r}{2}}\right)$$

$$\implies \tau_{\theta\theta} = 2\mu\left(\frac{\partial u_\theta}{r\delta\theta} + \frac{u_r}{r}\right)$$

$\tau_{zz} = 2\mu \times$ Rate of shear strain of Mid-Axial length of the element along Axial Direction

$$\implies \tau_{zz} = 2\mu\left(\frac{\left(u_z + \frac{\partial u_z}{\partial z}\frac{\delta z}{2} - \left(u_z - \frac{\partial u_z}{\partial z}\frac{\delta z}{2}\right)\right)}{\delta z}\right)$$

$$\implies \tau_{zz} = 2\mu\frac{\partial u_z}{\partial z}$$

Worth noting from the fact that since the angular rotation of the CV is 0 so all adjacent symmetrical stresses must be equal , so proceeding with the next six symmetrical stresses:-

$\tau_{r\theta} = \tau_{\theta r} = \mu\times$ Rate of shear strain (Mid-Radial length of the element along Azimuthal Direction + Mid-Azimuthal length of the element along Radial Direction)

$$\implies \tau_{r\theta} = \mu\left(\frac{\left(u_r + \frac{\partial u_r}{\partial\theta}\frac{\delta\theta}{2}\right) - \left(u_r - \frac{\partial u_r}{\partial\theta}\frac{\delta\theta}{2}\right)}{\left(r + \frac{\delta r}{2}\right)\delta\theta} + \left(\frac{\left(u_\theta + \frac{\partial u_\theta}{\partial r}\frac{\delta r}{2}\right)}{r + \delta r} - \frac{\left(u_\theta - \frac{\partial u_\theta}{\partial r}\frac{\delta r}{2}\right)}{r}\right)\frac{r + \frac{\delta r}{2}}{\delta r}\right)$$

$$\implies \tau_{r\theta} = \mu\left(\frac{\partial u_r}{\left(r + \frac{\delta r}{2}\right)\partial\theta} + \left(\frac{ru_\theta + r\frac{\partial u_\theta}{\partial r}\delta r - u_\theta\left(r + \delta r\right)}{r^2}\right)\frac{r}{\delta r}\right)$$

$$\implies \tau_{r\theta} = \mu\left(\frac{\partial u_r}{r\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$\tau_{z\theta} = \tau_{\theta z} = \mu\times$ Rate of shear strain (Mid-Axial length of the element along Azimuthal Direction + Mid-Azimuthal length of the element along Axial Direction)

$$\implies \tau_{z\theta} = \mu\left(\frac{\left(u_z + \frac{\partial u_z}{\partial\theta}\frac{\delta\theta}{2}\right) - \left(u_z - \frac{\partial u_z}{\partial\theta}\frac{\delta\theta}{2}\right)}{\left(r + \frac{\delta r}{2}\right)\delta\theta} + \frac{\left(u_\theta + \frac{\partial u_\theta}{\partial z}\frac{\delta z}{2}\right) - \left(u_\theta - \frac{\partial u_\theta}{\partial z}\frac{\delta z}{2}\right)}{\delta z}\right)$$

$$\implies \tau_{z\theta} = \mu\left(\frac{\partial u_z}{r\partial\theta} + \frac{\partial u_\theta}{\partial z}\right)$$

$\tau_{zr} = \tau_{rz} = \mu\times$ Rate of shear strain of (Mid-Axial length of the element along Radial Direction + Mid-Radial length of the element along Axial Direction)

$$\implies \tau_{zr} = \mu\left(\frac{\left(u_z + \frac{\partial u_z}{\partial r}\frac{\delta r}{2}\right) - \left(u_z - \frac{\partial u_z}{\partial r}\frac{\delta r}{2}\right)}{\delta r} + \frac{\left(u_r + \frac{\partial u_r}{\partial z}\frac{\delta z}{2}\right) - \left(u_r - \frac{\partial u_r}{\partial z}\frac{\delta z}{2}\right)}{\delta z}\right)$$

$$\implies \tau_{zr} = \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)$$

So now we have all the 9 stress components in terms of velocities as following:-

$$\tau_{rr} = 2\mu\frac{\partial u_r}{\partial r} \qquad \tau_{\theta\theta} = 2\mu\left(\frac{\partial u_\theta}{r\delta\theta} + \frac{u_r}{r}\right) \qquad \tau_{zz} = 2\mu\frac{\partial u_z}{\partial z}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(\frac{\partial u_r}{r\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \qquad \tau_{zr} = \tau_{rz} = \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) \qquad \tau_{\theta z} = \tau_{z\theta} = \mu\left(\frac{\partial u_z}{r\partial\theta} + \frac{\partial u_\theta}{\partial z}\right) \tag{16}$$

These relations (16) are the basic Newtonian Viscosity relations for cylindrical co-ordinate systems. Now we can set our eyes on the main Newtonian- Navier-Stokes equations for Cylindrical Co-ordinates.

4.4 Derivation of Dimensional N-S equation for a Newtonian Fluid in Cylindrical Domain

As we have already split the main equation into three separate momentum equations and the main focus area of the solving is the RHS equations, solving each equation separately, we get :-

- **RHS of Radial Momentum Equation (13) :-**

$$-\frac{\partial P}{\partial r} + \frac{1}{r}\frac{\partial(r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta} + \frac{\partial\tau_{zr}}{\partial z} - \frac{\tau_{\theta\theta}}{r}$$

$$\implies -\frac{\partial P}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\left(2\mu\frac{\partial u_r}{\partial r}\right)\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\mu\left(\frac{\partial u_r}{r\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)\right) - \frac{2\mu}{r}\left(\frac{\partial u_\theta}{r\delta\theta} + \frac{u_r}{r}\right)$$

$$\implies -\frac{\partial P}{\partial r}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\partial u_r}{\partial r}\right)\right)+\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}+\frac{\partial^2 u_r}{r^2\partial\theta^2}+\frac{1}{r}\frac{\partial^2 u_\theta}{\partial\theta\partial r}-\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial^2 u_r}{\partial z^2}+\frac{\partial^2 u_z}{\partial z\partial r}-\frac{2}{r^2}\frac{\partial u_\theta}{\delta\theta}-\frac{2u_r}{r^2}\right)$$

$$\implies -\frac{\partial P}{\partial r}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\partial u_r}{\partial r}\right)\right)-\frac{u_r}{r^2}+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right)+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)+\frac{\partial}{\partial r}\left(\frac{\partial u_z}{\partial z}\right)+\frac{\partial^2 u_r}{r^2\partial\theta^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\delta\theta}\right)$$

$$\implies -\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right)+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z}\right)+\frac{\partial^2 u_r}{r^2\partial\theta^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\delta\theta}\right)$$

By using the Conservation law of equation (4), the equation will be refined as :-

$$\implies -\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right)+\frac{\partial^2 u_r}{r^2\partial\theta^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\delta\theta}\right) \tag{17}$$

- **RHS of Azimuthal Momentum Equation (14) :-**

$$-\frac{1}{r}\frac{\partial P}{\partial\theta}+\frac{1}{r^2}\frac{\partial\left(r^2\tau_{r\theta}\right)}{\partial r}+\frac{\partial\tau_{z\theta}}{\partial z}+\frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta}$$

$$\implies -\frac{1}{r}\frac{\partial P}{\partial\theta}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\mu\left(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)\right)+\frac{1}{r}\frac{\partial}{\partial\theta}2\mu\left(\frac{\partial u_\theta}{r\delta\theta}+\frac{u_r}{r}\right)+\frac{\partial}{\partial z}\mu\left(\frac{\partial u_z}{r\partial\theta}+\frac{\partial u_\theta}{\partial z}\right)$$

$$\implies -\frac{1}{r}\frac{\partial P}{\partial\theta}+\mu\left(\frac{1}{r}\frac{\partial^2 u_r}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{1}{r}\frac{\partial^2 u_z}{\partial z\partial\theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right)$$

$$\implies -\frac{1}{r}\frac{\partial P}{\partial\theta}+\mu\left(\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)+\frac{\partial^2 u_\theta}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{2}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{1}{r}\frac{\partial^2 u_z}{\partial z\partial\theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right)$$

$$\implies -\frac{1}{r}\frac{\partial P}{\partial\theta}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(ru_\theta)}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z}\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right)$$

By using the Conservation law of equation (4), the equation will be refined as :-

$$\implies -\frac{1}{r}\frac{\partial P}{\partial\theta}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(ru_\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right) \tag{18}$$

- **RHS of Axial Momentum Equation (15) :-**

$$-\frac{\partial P}{\partial z}+\frac{1}{r}\frac{\partial(r\tau_{rz})}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta}+\frac{\partial\tau_{zz}}{\partial z}$$

$$\implies -\frac{\partial P}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\mu\left(\frac{\partial u_z}{r\partial\theta}+\frac{\partial u_\theta}{\partial z}\right)\right)+\frac{\partial}{\partial z}\left(2\mu\frac{\partial u_z}{\partial z}\right)$$

$$\implies -\frac{\partial P}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\partial u_z}{\partial r}\right)\right)+\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2}+\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)+\frac{\partial}{\partial z}\left(\frac{\partial u_z}{\partial z}\right)+\frac{\partial^2 u_z}{\partial z^2}\right)$$

$$\implies -\frac{\partial P}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\partial u_z}{\partial r}\right)\right)+\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z}\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right)$$

By using the Conservation law of equation (4), the equation will be refined as :-

$$\implies -\frac{\partial P}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\partial u_z}{\partial r}\right)\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right) \tag{19}$$

Using these sub equations (17), (18) & (19) in equations - (13), (14) & (15) :-

1. Radial Momentum Equation :-

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\theta^2}{r}\right)=-\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right)+\frac{\partial^2 u_r}{r^2\partial\theta^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\delta\theta}\right)$$

Dividing by ρ on both sides we get :-

$$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\theta^2}{r}=-\frac{1}{\rho}\frac{\partial P}{\partial r}+\nu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right)+\frac{\partial^2 u_r}{r^2\partial\theta^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\delta\theta}\right) \tag{20}$$

Similarly for other 2 equations as well,

2. Azimuthal Momentum Equation :-

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} = -\frac{1}{r\rho}\frac{\partial P}{\partial \theta} + \nu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (ru_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial z^2}\right) \tag{21}$$

3. Axial Momentum Equation :-

$$\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\partial u_z}{\partial r}\right)\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right) \tag{22}$$

4. Continuity Equation :-

$$\frac{1}{r}\frac{\partial (ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \tag{23}$$

These above 4 equations namely the Continuity, radial momentum, azimuthal momentum & axial momentum equations are known as cylindrical Navier Stokes Equation for Cylindrical Co-ordinate and provide assistance in solving the Pipe flow problems

5 Non-Dimensionalisation of the Cylindrical N-S Equations

The above 4 equations - (20), (21), (22) & (23) are in dimensional form that means they have units involved in their expression like meters/seconds for velocity, meters for length, kilograms for mass, etc. This is of less relevance to us to study the effect of magnitude of different components of expression like pressure, mass, volume flow rate or velocity we cant do it by these expression as mass cannot be compared with pressure and velocities so on and so forth. So it is but devised by scientist to first non-dimensionalise this expression with the known parameters like inlet velocity, length scale of the domain, mass in flow,etc. The various other quantities like pressure scale or time scale can be derived from their combined relations as well. Let us first see the various annotations which will ease our understanding :-

- **Length Scale (l_{ref}) :-** We choose the diameter (D) of the pipe as our length scale as it is generally the smallest known length in the cylindrical flow problem if no other obstruction is mentioned.

  $$L_{ref} = D$$

  Velocity Scale ($U_{\mathbf{ref}}$):
  Since the inlet velocity is typically known in fluid flow problems, we can select any one of the components of the velocity field at the inlet as our reference. The three main velocity components at the inlet are: axial (u_z), radial (u_r), and azimuthal (u_θ) velocities. Among these, the most significant is the **axial velocity** (u_z), as it plays a crucial role in both the mass flow rate and the kinetic energy of the flow.

  $$u_{\text{ref}} = u_{z_{\text{avg}}}$$

  Reason for choosing $u_{z_{\mathbf{avg}}}$:
  The primary reason for selecting the axial velocity as the reference is that it directly governs the mass flow rate at the inlet. The axial velocity component u_z is responsible for the flow of fluid into the domain, which is crucial for defining the mass flux. On the other hand, the radial and azimuthal components (u_r and u_θ) only contribute to the distribution of momentum and energy, rather than the mass flow. Hence, the axial velocity is the natural choice for defining the reference velocity in this context.

- **Time Reference t_{ref}:-**

  $$\because \text{time scale} = \frac{\text{Length Scale}}{\text{Velocity Scale}}$$

  . So ot time reference becomes as follows :-

  $$t_{ref} = \frac{l_{ref}}{U_{ref}}$$

- **Density Reference :-**SInce we have assubmed fluid to have a constant density throught out the flow as well as it is a single phase fluid and no phase change in the flow field so, the density refernce will be the density inlet :-

  $$\rho_{ref} = \rho_{inlet}$$

- **Pressure/Stress Reference :-** Te reference stress or pressure will be considered as the gauge pressure of fluid exterted at inlet boundary due to inflow :-

  $$p_{ref} = \frac{\textit{Fluid Force at inlet}}{\text{Area of Inlet Cross Section}}$$

  $$\implies P_{ref} = \frac{\rho_{ref} U_{ref}^2 \frac{\pi D^2}{4}}{\frac{\pi D^2}{4}}$$

  $$\implies P_{ref} = \rho_{ref} U_{ref}^2$$

Parameter	Dimensional Variable	Non- Dimensional Variable	Reference for Division
Axial Length	z	z^*	Diameter (D)
Radial Length	r	r^*	Diameter (D)
Azimuthal Angle	θ	θ	Angle (θ)
Axial Velocity	u_z	u_z^*	$U_{z_{avg}}$
Azimuthal Velocity	u_θ	u_θ^*	$U_{z_{avg}}$
Radial Velocity	u_r	u_r^*	$U_{z_{avg}}$
Time	t	t^*	$\frac{D}{U_{z_{avg}}}$
Density	ρ	ρ	ρ_{inlet}
Pressure	P	P^*	$\rho_{inlet} \times U_{z_{avg}}^2$

Table 3: **Table showing the conversions of various parameters in the governing equations from dimensional to non-dimensional forms**

The various scaled parameters and their choice discussion has been elaborated below :- Below is the relations between dimensional to non-dimensional forms of parameters :-

- Non-Dimensional Length :- $L^* = \frac{L}{L_{ref}}$

$$r = r^* = \frac{r}{D}, \qquad z^* = \frac{z}{D}$$

- Non-Dimensional Velocity :- $U^* = \frac{U}{U_{ref}}$

$$u_r^* = \frac{u_r}{U_{z_{avg}}}, \qquad u_\theta^* = \frac{u_\theta}{U_{z_{avg}}} \qquad u_z^* = \frac{u_z}{U_{z_{avg}}}$$

- Non-Dimensional Time :- $t^* = \frac{t}{t_{ref}}$

$$t^* = \frac{t}{\frac{D}{U_{z_{avg}}}} = t \times \frac{U_{z_{avg}}}{D}$$

- Non-Dimensional Pressure :- $P = P^* \times P_{ref}$

$$P^* = \frac{P}{\left(\rho_{ref} U_{z_{avg}}^2\right)}$$

- Non-Dimensional Density : $\rho = \rho_{inlet} = \rho$ ($\because$ Density is constant in the flow field)
- Non-Dimensional Angle : As by definition of $\theta = \dfrac{\text{Arc Length}}{\text{Radius of Cross-Section}}$, so it is already non-dimensional.

Now we begin with Non-Dimensionalisation the Governing Equations :-

1. **Continuity Equation :-**

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

$$\implies \frac{1}{r^*D}\frac{\partial(r^*Du_r^*U_{z_{avg}})}{\partial r^*D} + \frac{1}{r^*D}\frac{\partial u_\theta^* U_{z_{avg}}}{\partial \theta} + \frac{\partial u_z^* U_{z_{avg}}}{\partial z^*D} = 0$$

Multiplying by $\dfrac{D}{U_{z_{avg}}}$ as they are constants, we get :-

$$\boxed{\frac{1}{r^*}\frac{\partial(r^*u_r^*)}{\partial r^*} + \frac{1}{r^*}\frac{\partial u_\theta^*}{\partial \theta} + \frac{\partial u_z^*}{\partial z^*} = 0} \tag{24}$$

2. **Radial Momentum Equation :-**

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right) + \frac{\partial^2 u_r}{r^2\partial\theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\delta\theta}\right)$$

$$\implies \frac{\partial u_r^* U_{z_{avg}}}{\partial t^*\left(\frac{D}{U_{z_{avg}}}\right)} + u_r^* U_{z_{avg}}\frac{\partial u_r^* U_{z_{avg}}}{\partial r^* D} + \frac{u_\theta^* U_{z_{avg}}}{r^*D}\frac{\partial u_r^* U_{z_{avg}}}{\partial \theta} + u_z^* U_{z_{avg}}\frac{\partial u_r^* U_{z_{avg}}}{\partial z^* D} - \frac{(u_\theta^* U_{z_{avg}})^2}{r^*D} = -\frac{1}{\rho}\frac{\partial\left(P^*\rho U_{z_{avg}}^2\right)}{\partial r^*D} +$$

$$\nu\left(\frac{\partial}{\partial r^*D}\left(\frac{1}{r^*D}\frac{\partial(r^*Du_r^*U_{z_{avg}})}{\partial r^*D}\right) + \frac{\partial^2 u_r^* U_{z_{avg}}}{(r^*D)^2\partial\theta^2} + \frac{\partial^2 u_r^* U_{z_{avg}}}{\partial(z^*D)^2} - \frac{2}{(r^*D)^2}\frac{\partial u_\theta^* U_{z_{avg}}}{\delta\theta}\right)$$

Multiplying by $\frac{D}{U_{z_{avg}}^2}$ as they are constants, we get :-

$$\implies \frac{\partial u_r^*}{\partial t^*} + u_r^*\frac{\partial u_r^*}{\partial r^*} + \frac{u_\theta^*}{r^*}\frac{\partial u_r^*}{\partial \theta} + u_z^*\frac{\partial u_r^*}{\partial z^*} + \frac{(u_\theta^*)^2}{r^*} = -\frac{\partial P^*}{\partial r^*} + \frac{\nu}{DU_{z_{avg}}}\left(\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}\frac{\partial (r^* u_r^*)}{\partial r^*}\right) + \frac{\partial^2 u_r^*}{(r^*)^2 \partial \theta^2} + \frac{\partial^2 u_r^*}{\partial (z^*)^2} - \frac{2}{(r^*)^2}\frac{\partial u_\theta^*}{\delta \theta}\right)$$

As we basically know that $\frac{DU_{z_{avg}}}{\nu}$ is the most famous non-dimensional parameter in fluid studies and is known as the **Reynolds Number (Re)** .The Reynolds number represents the ratio of inertial forces to viscous forces in a fluid. It helps us cycle through the various properties of fluid without indulging into any single parameter alone. Flows with different Reynolds Numbers have different characteristics. So our equation now becomes :-

$$\boxed{\frac{\partial u_r^*}{\partial t^*} + u_r^*\frac{\partial u_r^*}{\partial r^*} + \frac{u_\theta^*}{r^*}\frac{\partial u_r^*}{\partial \theta} + u_z^*\frac{\partial u_r^*}{\partial z^*} + \frac{(u_\theta^*)^2}{r^*} = -\frac{\partial P^*}{\partial r^*} + \frac{1}{Re}\left(\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}\frac{\partial (r^* u_r^*)}{\partial r^*}\right) + \frac{\partial^2 u_r^*}{(r^*)^2 \partial \theta^2} + \frac{\partial^2 u_r^*}{\partial (z^*)^2} - \frac{2}{(r^*)^2}\frac{\partial u_\theta^*}{\delta \theta}\right)} \tag{25}$$

3. **Azimuthal Momentum Equation :-**

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} = -\frac{1}{r\rho}\frac{\partial P}{\partial \theta} + \nu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial z^2}\right)$$

$$\implies \frac{\partial u_\theta^* U_{z_{avg}}}{\partial t^* \frac{D}{U_{z_{avg}}}} + u_r^* U_{z_{avg}}\frac{\partial u_\theta^* U_{z_{avg}}}{\partial r^* D} + \frac{u_\theta^* U_{z_{avg}}}{r^* D}\frac{\partial u_\theta^* U_{z_{avg}}}{\partial \theta} + u_z^* U_{z_{avg}}\frac{\partial u_\theta^* U_{z_{avg}}}{\partial z^* D} + \frac{u_r^* U_{z_{avg}} u_\theta^* U_{z_{avg}}}{r^* D} = -\frac{1}{r^* D\rho}\frac{\partial P^* \rho U_{z_{avg}}^2}{\partial \theta} +$$
$$\nu\left(\frac{\partial}{\partial r^* D}\left(\frac{1}{r^* D}\frac{\partial (r^* D u_\theta^* U_{z_{avg}}^2)}{\partial r^* D}\right) + \frac{1}{(r^* D)^2}\frac{\partial^2 u_\theta^* U_{z_{avg}}^2}{\partial \theta^2} + \frac{2}{(r^* D)^2}\frac{\partial u_r^* U_{z_{avg}}^2}{\partial \theta} + \frac{\partial^2 u_\theta^* U_{z_{avg}}^2}{\partial (z^* D)^2}\right)$$

Multiplying by $\frac{D}{U_{z_{avg}}^2}$ as they are constants, we get :-

$$\implies \frac{\partial u_\theta^*}{\partial t^*} + u_r^*\frac{\partial u_\theta^*}{\partial r^*} + \frac{u_\theta^*}{r^*}\frac{\partial u_\theta^*}{\partial \theta} + u_z^*\frac{\partial u_\theta^*}{\partial z^*} + \frac{u_r^* u_\theta^*}{r^*} = -\frac{1}{r^*}\frac{\partial P^*}{\partial \theta} + \frac{\nu}{DU_{z_{avg}}}\left(\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}\frac{\partial (r^* u_\theta^*)}{\partial r^*}\right) + \frac{1}{(r^*)^2}\frac{\partial^2 u_\theta^*}{\partial \theta^2} + \frac{2}{(r^*)^2}\frac{\partial u_r^*}{\partial \theta} + \frac{\partial^2 u_\theta^*}{\partial (z^*)^2}\right)$$

$$\boxed{\frac{\partial u_\theta^*}{\partial t^*} + u_r^*\frac{\partial u_\theta^*}{\partial r^*} + \frac{u_\theta^*}{r^*}\frac{\partial u_\theta^*}{\partial \theta} + u_z^*\frac{\partial u_\theta^*}{\partial z^*} + \frac{u_r^* u_\theta^*}{r^*} = -\frac{1}{r^*}\frac{\partial P^*}{\partial \theta} + \frac{1}{Re}\left(\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}\frac{\partial (r^* u_\theta^*)}{\partial r^*}\right) + \frac{1}{(r^*)^2}\frac{\partial^2 u_\theta^*}{\partial \theta^2} + \frac{2}{(r^*)^2}\frac{\partial u_r^*}{\partial \theta} + \frac{\partial^2 u_\theta^*}{\partial (z^*)^2}\right)} \tag{26}$$

4. **Axial Momentum Equation :-**

$$\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\partial u_z}{\partial r}\right)\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right)$$

$$\implies \frac{\partial u_z^* U_{z_{avg}}}{\partial t^* \frac{D}{U_{z_{avg}}}} + u_r^* U_{z_{avg}}\frac{\partial u_z^* U_{z_{avg}}}{\partial r^* D} + \frac{u_\theta^* U_{z_{avg}}}{r^* D}\frac{\partial u_z^* U_{z_{avg}}}{\partial \theta} + u_z^* U_{z_{avg}}\frac{\partial u_z^* U_{z_{avg}}}{\partial z^* D} = -\frac{1}{\rho}\frac{\partial P^* \rho U_{z_{avg}}^2}{\partial z} +$$
$$\nu\left(\frac{1}{r^* D}\frac{\partial}{\partial r^* D}\left(r^* D\left(\frac{\partial u_z^* U_{z_{avg}}^2}{\partial r^* D}\right)\right) + \frac{1}{(r^* D)^2}\frac{\partial^2 u_r^* U_{z_{avg}}^2}{\partial \theta^2} + \frac{\partial^2 u_z^* U_{z_{avg}}^2}{\partial (z^* D)^2}\right)$$

Multiplying by $\frac{D}{U_{z_{avg}}^2}$ as they are constants, we get :-

$$\implies \frac{\partial u_z^*}{\partial t^*} + u_r^*\frac{\partial u_z^*}{\partial r^*} + \frac{u_\theta^*}{r^*}\frac{\partial u_z^*}{\partial \theta} + u_z^*\frac{\partial u_z^*}{\partial z^*} = -\frac{\partial P^*}{\partial z^*} + \frac{\nu}{DU_{z_{avg}}}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\left(\frac{\partial u_z^*}{\partial r^*}\right)\right) + \frac{1}{(r^*)^2}\frac{\partial^2 u_r^*}{\partial \theta^2} + \frac{\partial^2 u_z^*}{\partial (z^*)^2}\right)$$

$$\boxed{\frac{\partial u_z^*}{\partial t^*} + u_r^*\frac{\partial u_z^*}{\partial r^*} + \frac{u_\theta^*}{r^*}\frac{\partial u_z^*}{\partial \theta} + u_z^*\frac{\partial u_z^*}{\partial z^*} = -\frac{\partial P^*}{\partial z^*} + \frac{1}{Re}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\left(\frac{\partial u_z^*}{\partial r^*}\right)\right) + \frac{1}{(r^*)^2}\frac{\partial^2 u_r^*}{\partial \theta^2} + \frac{\partial^2 u_z^*}{\partial (z^*)^2}\right)} \tag{27}$$

Recalling the above 4 equations we get the :-

1. **Non-Dimensional Continuity Equation :-**

$$\boxed{\frac{1}{r^*}\frac{\partial(r^* u_r^*)}{\partial r^*} + \frac{1}{r^*}\frac{\partial u_\theta^*}{\partial \theta} + \frac{\partial u_z^*}{\partial z^*} = 0}$$

2. **Non-Dimensional Radial Momentum Equation :-**

$$\boxed{\frac{\partial u_r^*}{\partial t^*} + u_r^*\frac{\partial u_r^*}{\partial r^*} + \frac{u_\theta^*}{r^*}\frac{\partial u_r^*}{\partial \theta} + u_z^*\frac{\partial u_r^*}{\partial z^*} + \frac{(u_\theta^*)^2}{r^*} = -\frac{\partial P^*}{\partial r^*} + \frac{1}{Re}\left(\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}\frac{\partial(r^* u_r^*)}{\partial r^*}\right) + \frac{1}{(r^*)^2}\frac{\partial^2 u_r^*}{\partial \theta^2} + \frac{\partial^2 u_r^*}{\partial (z^*)^2} - \frac{2}{(r^*)^2}\frac{\partial u_\theta^*}{\delta\theta}\right)}$$

3. **Non-Dimensional Azimuthal Momentum Equation :-**

$$\boxed{\frac{\partial u_\theta^*}{\partial t^*} + u_r^*\frac{\partial u_\theta^*}{\partial r^*} + \frac{u_\theta^*}{r^*}\frac{\partial u_\theta^*}{\partial \theta} + u_z^*\frac{\partial u_\theta^*}{\partial z^*} + \frac{u_r^* u_\theta^*}{r^*} = -\frac{1}{r^*}\frac{\partial P^*}{\partial \theta} + \frac{1}{Re}\left(\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}\frac{\partial(r^* u_\theta^*)}{\partial r^*}\right) + \frac{1}{(r^*)^2}\frac{\partial^2 u_\theta^*}{\partial \theta^2} + \frac{\partial^2 u_\theta^*}{\partial (z^*)^2} + \frac{2}{(r^*)^2}\frac{\partial u_r^*}{\partial \theta}\right)}$$

4. **Non-Dimensional Axial Momentum Equation :-**

$$\boxed{\frac{\partial u_z^*}{\partial t^*} + u_r^*\frac{\partial u_z^*}{\partial r^*} + \frac{u_\theta^*}{r^*}\frac{\partial u_z^*}{\partial \theta} + u_z^*\frac{\partial u_z^*}{\partial z^*} = -\frac{\partial P^*}{\partial z^*} + \frac{1}{Re}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\left(\frac{\partial u_z^*}{\partial r^*}\right)\right) + \frac{1}{(r^*)^2}\frac{\partial^2 u_r^*}{\partial \theta^2} + \frac{\partial^2 u_z^*}{\partial (z^*)^2}\right)}$$

It is also clear that apart from the all the kinematic changes, our Kinematic Viscosity also is replaced by the Reynolds Number in the non-dimensional flow equations.

6 Boundary Conditions for the flow field

Although the Governing equations control the flow field parameters and gradients in their values, without the correct essential boundary conditions they remain helpless to predict the accurate flow field. It is the boundary conditions which decide the sudden changes or the value shifts as well as some definite pattern formation in the flow field. The Navier-Stokes equations (or any PDEs in fluid mechanics) have multiple possible solutions unless boundary conditions constrain them. BC's ensure a unique and physically realistic solution to the problem.

6.1 Types of Boundary Conditions :-

1. **Dirichlet Boundary Condition (Fixed Value Condition)** Specifies the exact value of the variable at the boundary.

$$u(x) = f(x) \quad \text{at} \quad x = x_0$$

   Example applications:

   - Fixed temperature in heat conduction: $T = T_0$.
   - No-slip velocity condition in fluid flow: $u = 0$ at a wall.

2. **Neumann Boundary Condition (Flux Condition)** Specifies the derivative (gradient) of the variable at the boundary.

$$\frac{du}{dx} = g(x) \quad \text{at} \quad x = x_0$$

   Example applications:

   - Heat flux condition: $-k\frac{dT}{dx} = q$.
   - Shear stress in fluid flow: $\tau = \mu\frac{du}{dy}$.

3. **Robin (Mixed) Boundary Condition** Combination of Dirichlet and Neumann conditions.

$$au + b\frac{du}{dx} = c \quad \text{at} \quad x = x_0$$

   Example applications:

   - Convective heat transfer: $-k\frac{dT}{dx} = h(T - T_\infty)$.
   - Radiative heat loss conditions.

4. **Periodic Boundary Condition** The variable at one boundary matches the value at the opposite boundary.

$$u(0) = u(L)$$

Example applications:

- Fully developed pipe flow in CFD.
- Simulating infinite domains using a small computational domain.

5. **Symmetry Boundary Condition** No flux (zero derivative) across a symmetry plane.

$$\frac{du}{dx} = 0 \quad \text{at} \quad x_{\text{symmetry}}$$

Example applications:

- Heat conduction problems with symmetric geometry.
- Flow around an object where only half the domain is modeled.

6. **Free Surface / Open Boundary Condition** Allows fluid or waves to exit the domain smoothly.

$$\frac{\partial u}{\partial x} = 0 \quad \text{at outlet}$$

Example applications:

- Air-water interface in CFD.
- Outlet in aerodynamic simulations.

These are the various types of boundary conditions available. Now for some specific problems like pipe flows.

6.2 Boundary Conditions for any arbitrary case for flow through a pipe problem

Here we will discuss in details of the boundary conditions that must exist in pipe boundaries to get the actual and accurate flow field results prediction.

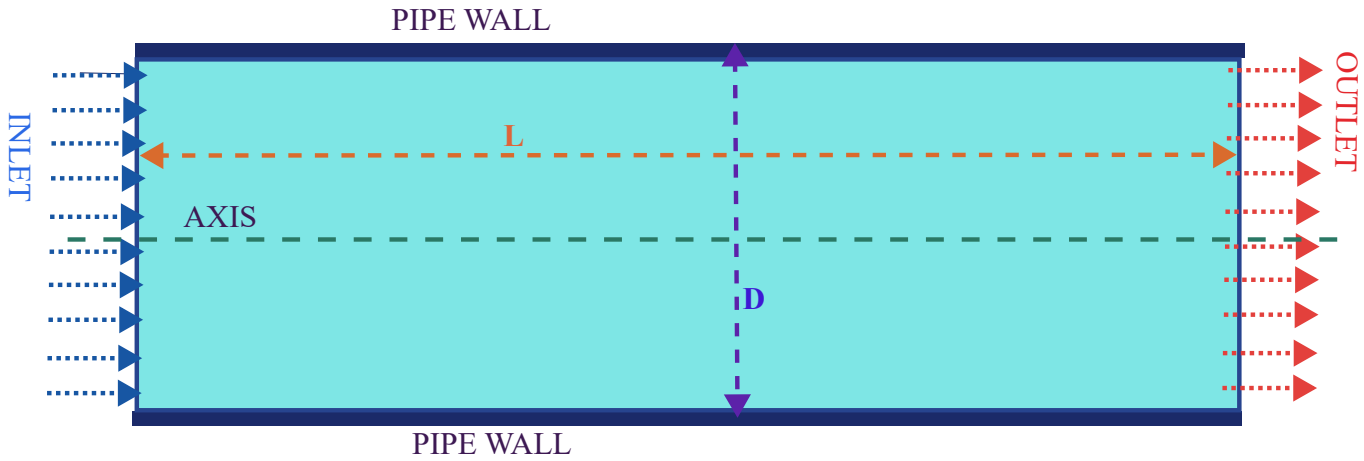


Figure 6: **The longitudinal view of the pipe flow describing the various boundary locations and the flow geometry.**

1. **INLET :-** It is the boundary from where the flow is entering or mass inlet in the pipe. It is shown as left boundary in the figure above and fluid is getting entry from this region in the flow domain. There can be in general two types of boundary conditions at the inlet - Pressure inlet where the entry pressure is known and velocity inlet where either the mass flow rate and area is known or the velocities at inlet i known meaning all the three cylindrical components of velocity namely - radial u_r , axial u_z and azimuthal u_θ. In the first case if the inlet pressure is fixed then the outlet pressure also needs to be known for the flow to take place. But in the second case has flexibility at outlet so we prefer the second case is preferred. It can have several cases :

- **Constant Axial Flow ;-** In this case first for a few axial length the the flow will develop to a Hagen-Poiseuille flow till when the maximum velocity becomes twice the inlet average velocity and after wards the flow will follow the same developed profile throughout the axial length.

For Dimensional Flow Fields	For Non-Dimensional Flow Fields
$u_r = 0; \quad u_\theta = 0; \quad u_z = \text{Constant value}(1, 2, 3...)$	$u_r^* = 0; \quad u_\theta^* = 0; \quad u_z^* = \text{Constant value (1,2,3...)}$

- **Parabolic Axial Flow ;-** Here an already developed flow profile is inlet into the system and this profile does not show much variation. The inlet velocity varies with the radius parabolically and the ratio of U_{max} to U_{avg} is $\frac{U_{max}}{U_{avg}} = 2$

For Dimensional Flow Fields :-

$$u_r = 0; \quad u_\theta = 0; \quad u_z = U_{max}\left(1 - \frac{r^2}{R^2}\right)$$

For Non-Dimensional Flow Fields :-

$$u_r^* = 0; \quad u_\theta^* = 0; \quad u_z^* = U_{z_{avg}}\left(1 - \frac{r^2}{R^2}\right)$$

- **Swirling Inlet Flow :-** This flow shows large variation axial, Azimuthal and radial profile plots and also witnesses the onset of instability regions in the flow field for higher Reynolds numbers.

For Dimensional Flow Fields :-

$$u_r = 0; \quad u_\theta = \omega.r; \quad u_z = U_{max}\left(1 - \frac{r^2}{R^2}\right)$$

For Non-Dimensional Flow Fields :-

$$u_r^* = 0; \quad u_\theta^* = \omega\left(\frac{D}{U_{z_{avg}}}\right).r^*; \quad u_z^* = U_{z_{avg}}\left(1 - \frac{r^2}{R^2}\right)$$

As you may notice from the above examples that it becomes extremely necessary to define the axial velocity at inlet very properly because only the axial velocity causes external mass of fluid to enter the system and causing the flow field to develop and show changes.

2. **WALLS OF THE PIPE :-** In continuum it is considered that fluid becomes relatively stationary to the walls, i.e., it sticks to it and 0 relative motion. Main requirement is that the walls must be no slip and impermeable boundaries So:-

Wall Condition	Dimensional Flow Fields	Non-Dimensional Flow Fields
Stationary	$(u_r)_{fw} = (u_\theta)_{fw} = (u_z)_{fw} = 0$	$(u_r^*)_{fw} = (u_\theta^*)_{fw} = (u_z^*)_{fw} = 0$
Moving	$(u_r)_{fw} = U_{r_{walls}};\ (u_\theta)_{fw} = U_{\theta_{walls}};\ (u_z)_{fw} = U_{z_{walls}}$	$(u_r^*)_{fw} = U^*_{r_{walls}};\ (u_\theta^*)_{fw} = U^*_{\theta_{walls}};\ (u_z^*)_{fw} = U^*_{z_{walls}}$

Ideally no pressure boundary condition is given at the walls since majority of modern solver rely on Pressure-Velocity coupling and specifying both at boundaries will make the solution over constraint and cause problems in convergence. But if still one is wiling then Neumann condition that is normal pressure gradient into the wall must be 0. $\frac{\partial P}{\partial n} = 0$

3. **OUTLET :-** It is that part of boundary from where the fluid is moving out of the control volume,i.e., there is no shear stress involved and fluid is exiting with no net force. Therefore at outlet one must equate the shear stresses on the surface to 0 while the normal stress must be equal to the exit pressure, meaning a Dirichlet boundary condition for pressure must be exist at the outlet or pressure must be quantified at the outlet.

$$\tau_{zr} = \tau_{z\theta} = 0 \qquad \tau_{zz} = p_{outlet}$$

For dimensional units :-

$$\implies \tau_{zr} = \nu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = 0$$

$$\implies \tau_{z\theta} = \nu\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) = 0$$

$$\implies \tau_{zz} = 2\nu\frac{\partial u_z}{\partial z} = p$$

On keeping the exit gauge pressure $P_{outlet} = 0$; saves our efforts for genuine cases where exit is to atmospheric pressures :-

$$\implies \frac{\partial u_z}{\partial z} = 0$$

$$\implies \frac{\partial \tau_{z\theta}}{\partial z} = \nu\left(\frac{\partial^2 u_\theta}{\partial z^2} + \frac{\partial^2 u_z}{\partial z \partial \theta}\right) = 0 \implies \frac{\partial^2 u_\theta}{\partial z^2} = 0$$

$$\implies \frac{\partial \tau_{zr}}{\partial z} = \nu\left(\frac{\partial^2 u_r}{\partial z^2} + \frac{\partial^2 u_z}{\partial z \partial r}\right) = 0 \implies \frac{\partial^2 u_r}{\partial z^2} = 0$$

These are essential Outlet BC's for dimensional flows.

For non-dimensional units :-

$$\implies \tau^*_{zr} = \frac{1}{Re}\left(\frac{\partial u_r^*}{\partial z^*} + \frac{\partial u_z^*}{\partial r^*}\right) = 0$$

$$\implies \tau^*_{z\theta} = \frac{1}{Re}\left(\frac{\partial u_\theta^*}{\partial z^*} + \frac{1}{r^*}\frac{\partial u_z^*}{\partial \theta}\right) = 0$$

$$\implies \tau^*_{zz} = 2\frac{1}{Re}\frac{\partial u_z^*}{\partial z^*} = p^*$$

On keeping the exit gauge pressure $P^*_{outlet} = 0$; saves our efforts for genuine cases where exit is to atmospheric pressures :-

$$\implies \frac{\partial u_z^*}{\partial z^*} = 0$$

$$\implies \frac{\partial \tau^*_{z\theta}}{\partial z^*} = \frac{1}{Re}\left(\frac{\partial^2 u_\theta^*}{\partial (z^*)^2} + \frac{\partial^2 u_z^*}{\partial z^* \partial \theta}\right) = 0 \implies \frac{\partial^2 u_\theta^*}{\partial (z^*)^2} = 0$$

$$\implies \frac{\partial \tau^*_{zr}}{\partial z^*} = \frac{1}{Re}\left(\frac{\partial^2 u_r^*}{\partial (z^*)^2} + \frac{\partial^2 u_z^*}{\partial z^* \partial r^*}\right) = 0 \implies \frac{\partial^2 u_r^*}{\partial (z^*)^2} = 0$$

These are essential Outlet BCs for non-dimensional flows.

4. **AXIS OF SYMMETRY :-** It means there is an axis of symmetry in the flow field and the flow is symmetrical about this boundary. Since the flow field is symmetrical so no mass or energy crosses this boundary so normal derivatives will be zero and it is generally z axis and perpendicular direction is radial for cross flows ,i.e.,

These were the essential boundary conditions for the fluid flow through a cylindrical pipe.

For Dimensional Flow Fields	For Non-Dimensional Flow Fields
$\frac{\partial u_r}{\partial r} = \frac{\partial u_\theta}{\partial r} = \frac{\partial u_z}{\partial r} = \frac{\partial P}{\partial r} = 0$	$\frac{\partial u_r^*}{\partial r^*} = \frac{\partial u_\theta^*}{\partial r^*} = \frac{\partial u_z^*}{\partial r^*} = \frac{\partial P^*}{\partial r^*} = 0$

7 References

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